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Quaternary Modular Summations

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A fundamental isometry in the theory of modular forms [1] is extended to a quaternary generalization of the modular group.

A quaternary number is an expression of the form $d + ia + jb + kc$ where a, b, c, and d are real numbers. The sum of the quaternary numbers $d_1 + ia_1 + jb_1 + kc_1$ and $d_2 + ia_2 + jb_2 + kc_2$ is the quaternary number $d + ia + jb + kc$ where $a = a_1 + a_2$, $b = b_1 + b_2$, $c = c_1 + c_2$, and $d = d_1 + d_2$. The product $(d_1 + ia_1 + jb_1 + kc_1) \times (d_2 + ia_2 + jb_2 + kc_2)$ is the quaternary number $d + ia + jb + kc$ where

$$a = d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2$$

and

$$b = d_1b_2 + b_1d_2 + c_1a_2 - a_1c_2$$

and

$$c = d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2$$

and

$$d = d_1d_1 - a_1a_2 - b_1b_2 - c_1c_2 .$$

The conjugate of the quaternary number $w = d + ia + jb + kc$ is $\bar{w} = d - ia - jb - kc$. The absolute value $|w|$ of a quaternary number w is the nonnegative number whose square is $w\bar{w}$.

A quaternary integer is a quaternary number w such that the real numbers $\bar{w}i - iw$, $\bar{w}j - jw$, $\bar{w}k - kw$, $w + \bar{w}$, and $w\bar{w}$ are integers. Sums and products of quaternary integers are quaternary integers. A quaternary integer is said to be a unit if it has absolute value one. There are 24 quaternary units. A quaternary integer a is said to be a left divisor of a quaternary integer b if $b = ac$ for a quaternary integer c. If a and b are quaternary integers, not both zero, an element of least absolute value in the set of nonzero quaternary integers of the form $a\bar{d} + b\bar{c}$ for quaternary integers c and d is a left divisor of a and b. Such a divisor is called a greatest common left divisor of a and b. A greatest common left divisor of a and b is unique within a unit factor on the right. A quaternary integer n has a quaternary integer s as a left divisor if $s\bar{s}$ is two and $n\bar{n}$ is even. If r is a positive integer, let $\rho(r)$ be the product of r and the numbers of the form $1 + 1/p$, $\varphi(r)$ the product of r and the numbers of the form $1 - 1/p$, where p is an odd prime divisor of r. For every positive integer r the number of quaternary integers n, not divisible by any positive integer other than one, such that $n\bar{n} = r$ is equal to $24\,\rho(r)$.

If r is a positive integer, two quaternary integers a and b are said to be congruent modulo r, $a \equiv b$ modulo r, if r is a divisor of $b - a$. Addition, multiplication, and conjugation of quaternary integers modulo r are defined by reference to corresponding quaternary integers. The number of quaternary integers modulo r is r^4. Let C be a nonzero quaternary integer, let s be the greatest positive divisor of C, and let $r = C\bar{C}$. If r is odd, the number of quaternary integers modulo r of the form Cn for a quaternary integer n is $(r/s^2)^2$. The number of quaternary integers D such that $C\bar{D} + D\bar{C} \equiv 0$ modulo r and such that one is a greatest common left divisor of C and D is $(r/s^2)^2\, s^2\varphi(r/s^2)\,\varphi(s)$.

Let Γ be the set of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ whose entries are quaternary integers and which have an inverse of the form $r^{-1}\left(\begin{smallmatrix} \bar{D} & \bar{B} \\ \bar{C} & \bar{A} \end{smallmatrix}\right)$ for a positive integer r, called the quaternary determinant of the matrix. The product of two elements of Γ is an element of Γ whose quaternary determinant is the product of the quaternary determinants of its factors. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is an element of Γ whose entries commute with each other, then $\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ is an element of Γ of equal quaternary determinant. The quaternary modular group is the set of elements of Γ of quaternary determinant one. If a, b, c, and d are quaternary integers such that $a\bar{b} + b\bar{a} = 0$, such that one is a greatest common left divisor of a and b, such that $c\bar{d} + d\bar{c} = 0$, and such that one is a greatest common left divisor of c and d, then

$$(c, d) = (a, b)\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for an element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the quaternary modular group. If

$$\begin{pmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{pmatrix}$$

are elements of Γ of equal quaternary determinant and if the entries of neither matrix have a nontrivial common positive divisor, then the equation

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{pmatrix} = \begin{pmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

is soluble for elements

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

of the quaternary modular group.

The quaternary right half-space is the set of quaternary numbers z such that $z + \bar{z} > 0$. The quaternary symplectic group is the set of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with quaternary entries which have an inverse of the form $\left(\begin{smallmatrix} \bar{D} & \bar{B} \\ \bar{C} & \bar{A} \end{smallmatrix}\right)$. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is an element of the group, the transformation $z \to (Az + B)/(Cz + D)$ maps the right half-space onto itself. If $F(z)$ is a measurable real valued function of z in the half-space, then so is

$$G(z) = F((Az + B)/(Cz + D))$$

and

$$\int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} | F(t + iu + jv + kw)|^2\, t^{-4}\, du\, dv\, dw\, dt$$
$$= \int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} | G(t + iu + jv + kw)|^2\, t^{-4}\, du\, dv\, dw\, dt.$$

A self-adjoint transformation which commutes with the elements of the substitution group is given by $F(z) \to G(z)$ where

$$G(t + iu + jv + kw)$$
$$= -t^2(\partial^2F/\partial t^2 + \partial^2F/\partial u^2 + \partial^2F/\partial v^2 + \partial^2F/\partial w^2) + 2t(\partial F/\partial t).$$

Two points z and w in the quaternary right half-space are said to be equivalent with respect to the quaternary modular group if $w = (Az + B)/(Cz + D)$ for an element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the group. Two points z and w in the half-space are said to be symmetric with respect to the group if $\bar{w} = (Az + B)/(Cz + D)$ for an element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the group which is equal to $\left(\begin{smallmatrix} -\bar{D} & \bar{B} \\ \bar{C} & -\bar{A} \end{smallmatrix}\right)$. The set of points z in the right half-space which are not self-symmetric with respect to the group is the union of its connected components. A fundamental region for a normal subgroup of index three is obtained by piecing together eight adjacent components. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is an element of the group which is equal to $\left(\begin{smallmatrix} -\bar{D} & \bar{B} \\ \bar{C} & -\bar{A} \end{smallmatrix}\right)$, the set of points z in the half-space such that $\bar{z} = (Az + B)/(Cz + D)$ is the quaternary half-plane $Az + \overline{zA} = -B$ if C is zero and is the quaternary half-sphere $| Cz + D | = 1$ if C is nonzero. An example of a symmetric component is the set of points z in the right half-space which satisfy the inequalities $0 < \bar{z}i - iz < 1$, $0 < \bar{z}j - jz < 1$, $0 < \bar{z}k - kz < 1$, and $| z | > 1$.

THEOREM. *Let Ω be a fundamental region for the quaternary modular group. Let $\Phi(z)$ be a measurable real valued function of z in the right half-space such that $\Phi(z) = \Phi((Az + B)/(Cz + D))$ for every element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the quaternary modular group, such that*

$$\iiiint_{\Omega} | \Phi(t + iu + jv + kw)|^2\, t^{-4}\, du\, dv\, dw\, dt < \infty,$$

and such that

$$\iiiint_\Omega \Phi(t + iu + jv + kw)\, t^{-4}\, du\, dv\, dw\, dt = 0.$$

Then the inequality

$$\int_0^\infty \left| \int_0^1\int_0^1\int_0^1 \Phi(t + iu + jv + kw)\, du\, dv\, dw \right|^2 t^{-4}\, dt$$

$$\leqslant \tfrac{3}{2} \iiiint_\Omega |\, \Phi(t + iu + jv + kw)|^2\, t^{-4}\, du\, dv\, dw\, dt$$

is satisfied. If equality holds, the identity

$$\int_0^\infty \int_0^1\int_0^1\int_0^1 \Phi(t + iu + jv + kw)\, du\, dv\, dw$$

$$\times \int_0^1\int_0^1\int_0^1 \Psi(t + iu + jv + kw)\, du\, dv\, dw\, t^{-4}\, dt$$

$$= \tfrac{3}{2} \iiiint_\Omega \Phi(t + iu + jv + kw)\, \Psi(t + iu + jv + kw)\, t^{-4}\, du\, dv\, dw\, dt$$

holds for every measurable real valued function $\Psi(z)$ of z in the right half-space such that $\Psi(x) = \Psi((Az + B)/(Cz + D))$ for every element $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the quaternary modular group, such that

$$\iiiint_\Omega |\, \Psi(t + iu + jv + kw)|^2\, t^{-4}\, du\, dv\, dw\, dt < \infty,$$

and such that

$$\iiiint_\Omega \Psi(t + iu + jv + kw)\, t^{-4}\, du\, dv\, dw\, dt = 0.$$

Equality holds if the identity holds whenever

$$0 = \int_0^1\int_0^1\int_0^1 \Psi(t + iu + jv + dw)\, du\, dv\, dw.$$

A necessary and sufficient condition for equality is that the representation

$$24\Phi(z) = \sum \int_0^1\int_0^1\int_0^1 \Phi(|\, Cz + D\,|^{-2}\, (z + \bar{z})/2 + \imath u + jv + kw)\, du\, dv\, dw$$

holds with summation over all pairs of quaternary integers C and D such that $C\bar{D} + D\bar{C} = 0$ and such that one is a greatest common left divisor of C and D.

Let $h(t)$ be a measurable real valued function of $t > 0$ such that

$$\int_0^\infty | h(t)|^2 \, t^{-4} \, dt < \infty.$$

A necessary and sufficient condition that it be of the form

$$h(t) = \int_0^1 \int_0^1 \int_0^1 \Phi(t + iu + jv + kw) \, du \, dv \, dw$$

for some such function $\Phi(z)$ is that

$$(2^{-iz} - 2^{-1+iz}) \, \pi^{-\frac{3}{4}+\frac{1}{2}iz} \Gamma(\tfrac{3}{4} - \tfrac{1}{2}iz) \, \zeta(\tfrac{3}{2} - iz)$$

$$\times \, (1 - 2iz) \, \pi^{-\frac{1}{2}+iz} \Gamma(\tfrac{1}{2} - iz) \, \zeta(1 - 2iz) \int_0^\infty h(t) \, t^{-2\frac{1}{2}-iz} \, dt$$

be an even function of real z.

Proof. The argument is similar to the proof of [1, Theorem 6]. Let $h(t)$ be a bounded measurable real valued function of positive t, which vanishes for small t, such that

$$\int_0^\infty h(t) \, t^{-4} \, dt = 0.$$

Define the function $\Phi(z)$ in the right half-space by

$$24\Phi(z) = \sum h(| Cz + D |^{-2} (z + \bar{z})/2)$$

where summation is over all pairs of quaternary integers C and D such that $C\bar{D} + D\bar{C} = 0$ and such that one is a greatest common left divisor of C and D. Since only a finite number of terms are nonzero, an interchange of summation and integration yields

$$\int_0^1 \int_0^1 \int_0^1 \Phi(t + iu + jv + kw) \, du \, dv \, dw = h(t) + t^3 \int_0^\infty k(x) \, h(t^{-1}x^{-1}) \, dx$$

where

$$k(x) = 2\pi \sum_{n<x} \sum_{n=1}^{\infty} w(n) \, n^{-\frac{3}{2}} (x - n)^{\frac{1}{2}}$$

for a function $w(n)$ of positive integral n such that $w(ab) = w(a) \, w(b)$ whenever a and b are relatively prime, such that

$$w(n) = \sum_{k^2 | n} \rho(n/k^2) \, \varphi(n/k^2) \, k^2 \varphi(k)$$

when n is odd, and such that $w(2^{2n+1}) = 3 \times 2^{3n}$ and $w(2^{2n+2}) = 4 \times 2^{3n}$ for every positive integer n. Since

$$\int_n^\infty (t-n)^{\frac{1}{2}}\, t^{-\frac{3}{2}+iw}\, dt = \tfrac{1}{2}\pi^{\frac{1}{2}} n^{iw} \Gamma(-iw)/\Gamma(\tfrac{3}{2}-iw)$$

when $\mathrm{Re}(-iw) > 0$,

$$\begin{aligned}\int_0^\infty & k(t)\, t^{-\frac{3}{2}+iw}\, dt \\ &= \pi^{\frac{3}{2}}(1+2^{1+2iw})/(1-2^{-1+2iw}) \times \zeta(-\tfrac{1}{2}-iw)/\zeta(\tfrac{3}{2}-iw) \\ &\quad \times \zeta(-2iw)/\zeta(1-2iw) \times \Gamma(-iw)/\Gamma(\tfrac{3}{2}-iw)\end{aligned}$$

when $\mathrm{Re}(-iw) > 1$. By Euler's identity for the zeta function,

$$\int_0^\infty k(t)\, t^{-\frac{3}{2}+iw}\, dt = E(-w)/E(w)$$

is a function analytic in the upper half-plane except for a simple pole at $\frac{3}{2}i$. It is continuous in the closed half-plane, bounded by one on the real axis, and of bounded type and of nonpositive mean type in the upper half-plane. Since

$$\int_0^\infty h(t)\, t^{-2\frac{1}{2}+iw}\, dt$$

vanishes at $\frac{3}{2}i$,

$$\int_0^\infty k(t)\, t^{-\frac{3}{2}i+iw}\, dt \times \int_0^\infty h(t)\, t^{-2\frac{1}{2}+iw}\, dt$$

is analytic in the upper half-plane. The function

$$t^3 \int_0^\infty k(x)\, h(x^{-1}t^{-1})\, dx$$

vanishes for large t and

$$\int_0^\infty \left| t^3 \int_0^\infty k(x)\, h(x^{-1}t^{-1})\, dx \right|^2 dt \leqslant \int_0^\infty |h(t)|^2\, t^{-4}\, dt.$$

It follows that

$$\int_0^\infty \left| \int_0^1\int_0^1\int_0^1 \Phi(t+iu+jv+kw)\, du\, dv\, dw \right|^2 dt \leqslant 4 \int_0^\infty |h(t)|^2\, t^{-4}\, dt.$$

Since

$$3 \iiiint_{\Omega} | \Phi(t + iu + jv + kw)|^2 \, t^{-4} \, du \, dv \, dw \, dt$$

$$= \int_0^\infty \int_0^1 \int_0^1 \int_0^1 \Phi(t + iu + jv + kw) \, \bar{h}(t) \, t^{-4} \, du \, dv \, dw \, dt,$$

it follows that

$$3 \iiiint_{\Omega} | \Phi(t + iu + jv + kw)|^2 \, t^{-4} \, du \, dv \, dw \, dt \leqslant 2 \int_0^\infty | h(t)|^2 \, t^{-4} \, dt.$$

The remainder of the argument is similar to the proof of Theorem 6 of [1].

Reference

1. L. DE BRANGES, Coefficients of modular forms, *J. Math. Anal. Appl.*, to appear.